

THE FRACTIONAL FIXING NUMBER OF GRAPHS

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ABSTRACT. An automorphism group of a graph G is the set of all permutations of the vertex set of G that preserve adjacency and non-adjacency of vertices in G . A fixing set of a graph G is a subset of vertices of G such that only the trivial automorphism fixes every vertex in S . Minimum cardinality of a fixing set of G is called the fixing number of G . In this article, we define a fractional version of the fixing number of a graph. We formulate the problem of finding the fixing number of a graph as an integer programming problem. It is shown that a relaxation of this problem leads to a linear programming problem and hence to a fractional version of the fixing number of a graph. We also characterize the graphs G with the fractional fixing number $\frac{|V(G)|}{2}$ and the fractional fixing number of some families of graphs is also obtained.

1. MOTIVATION AND BACKGROUND

Motivation behind the development of the fractional idea has multiple aspects. One of the interesting aspect is that the fractional version multiplies the range of applications in operation research, scheduling or in various kind of assignment problems. Theorems in their fractional version are mostly easier to prove. Mostly for fractional and classical coefficients of graphs bounds are same or it may form conjecture in fractional version. Most of the times, the conjecture becomes refined theorem in their fractional version. Fractional version of parameters have drawn the attention of researchers to a wealth of new problems and conjectures. Fractional graph theory has modified the concept of integer-valued graph theory to the non-integral values.

Interesting aspects of fractional graph theory motivated Hedetniemi *et al.* to introduce the concept of the fractional domination number of a graph by linear relaxation of the integer programming problem of domination number of graphs [18]. A variety of work has been done on the fractional domination

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number of graphs, see [14, 23, 24, 27]. Currie *et al.* defined the fractional metric dimension of a graph as the optimal solution of the linear relaxation of the integer programming problem of the metric dimension of graphs [8]. The fractional metric dimension of graphs and graph products has also been studied [1, 11, 12, 13, 22, 28]. The metric dimension of a graph is an upper bound for the fixing number of graph [9]. Their relationship has been studied in [4, 9]. In this paper, we introduce fractional version of the fixing number of graph by introducing the idea of fixed graph. In the following paragraph, we introduce some relevant terminology needed for exposition of this idea.

All graphs considered in this paper are simple, non-trivial and undirected. Let $G = (V(G), E(G))$, when there is no ambiguity, we simply write $G = (V, E)$. The number of vertices and edges of G are called the *order* and the *size* of G respectively. For $u, v \in V(G)$, $u \sim v$ means u and v are adjacent and $u \not\sim v$ means u and v are not adjacent. The *open neighborhood* of a vertex u is $N_G(u) = \{v \in V(G) : v \sim u \text{ in } G\}$ and the *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. The number $|N_G(v)|$ is called the degree of v in G . The *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Two distinct vertices u and v in a graph G are said to be *twins* if $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$. A set $U \subseteq V(G)$ is called a *twin-set* of G if u, v are twins in G for every pair of distinct vertices $u, v \in U$.

A *permutation* of a set is a bijection from the set to itself. An *automorphism* of a graph is a permutation of the vertex set that preserves adjacency and non-adjacency of the vertices. An equivalent definition is: $\pi : V(G) \rightarrow V(G)$ is an automorphism of a graph G if for all $u, v \in V(G)$, $\pi(u) \sim \pi(v)$ if and only if $u \sim v$. The set of all automorphisms of G forms a group, called the *automorphism group* of the graph G . We use $\Gamma(G)$, or Γ if G is clear from the context, to denote the full automorphism group of a graph G . We consider the full automorphism group Γ acting on the vertex set V of G . For $u \in V$, the *orbit* $O(u)$ and *stabilizer* Γ_u of u is defined as $O(u) = \{\pi(u) : \pi \in \Gamma\}$ and $\Gamma_u = \{\pi \in \Gamma : \pi(u) = u\}$. For $T \subseteq V$, $\Gamma_T = \cap_{u \in T} \Gamma_u$. For $x \in V$, the subgroup Γ_x has a natural action on V and the orbit of u under this action is denoted by $O_x(u)$ i.e., $O_x(u) = \{\pi(u) : \pi \in \Gamma_x\}$. Define

$$A(G) = \{u : |O(u)| \geq 2\} \quad \text{and} \quad C(G) = \{u : |O(u)| = 1\}.$$

Then $V(G)$ is the disjoint union of $A(G)$ and $C(G)$. Define

$$V_a(G) = \{(u, v) \in A(G) \times A(G) : O(u) = O(v), u \neq v\}.$$

If G is a rigid graph (a graph with $\Gamma(G) = id$), then $V_a(G) = \emptyset$.

A set $\mathcal{D} \subseteq V(G)$ is called a *determining set* of G if whenever $\alpha, \beta \in \Gamma(G)$ such that $\alpha(v) = \beta(v)$ for all $v \in \mathcal{D}$, then $\alpha(u) = \beta(u)$ for every $u \in V(G)$. The *determining number* of a graph G is the order of a smallest determining set, denoted by $Det(G)$. Determining sets of graphs were introduced by Boutin in [2]. She gave several ways of finding and verifying determining sets. The natural lower bounds on the determining number of some graphs were also given. Determining sets are frequently used to identify the automorphism group of a graph. For further work on determining sets and its relation with other parameters, see [2, 4]. Erwin and Harary independently introduced an equivalent concept: the fixing number of a graph G [9]. A set $S \subset V$ is a *fixing set* of G if Γ_S is trivial, i.e., the only automorphism that fixes all vertices of S is the trivial automorphism. The cardinality of a smallest fixing set is called the *fixing number* of G , denoted by $fix(G)$. The fixing number of graphs is also used to study the symmetry of graphs and the relationships of groups and graphs [15]. The equivalence of determining and fixing set of graphs was also established in [15].

A fixing set S of G is minimal if no proper subset of S is a fixing set of G . In families of graphs like path, cycle, complete graph and complete bipartite graphs minimum fixing sets and minimal fixing sets have same cardinality. This is not the case always. For example, let C_{4n} , $n \geq 2$, be the cycle graph with $V(C_{4n}) = \{v_1, v_2, \dots, v_{4n}\}$. Attach two pendent vertices with v_1 and v_{2n+1} . Let us denote the resulting graph by \mathcal{C} . It is easy to check that $\{v_2\}$ and $\{v_{n+1}, v_{2n+1}\}$ are minimal fixing sets of \mathcal{C} . In this example, there exists a minimal fixing set whose cardinality is different from that of a minimum fixing set. We call the maximum cardinality of a minimal fixing set of a graph G , the upper fixing number of G , denoted by $fix^+(G)$. Note that $fix(G) \leq fix^+(G)$.

This paper is organized as follows: in section 2, we define the fixing neighborhood of a graph and the fixed graph. In this section fractional version of the fixing number of a graph is defined. We give an integer programming problem for the problem of finding the fixing number of a graph. We also show that by relaxing conditions of this problem, a linear programming problem is formulated which is a fractional version of the fixing number of a graph. In section 3, we characterize the graphs G with the fractional fixing number $\frac{|V(G)|}{2}$. Section 4 of this paper is devoted to the study of the fractional fixing number of some families of graphs. In section 5, we study the fractional fixing number

of corona product of graphs. In the last section, we study the fractional fixing number of composition product of graphs.

2. FIXING NEIGHBORHOOD AND FIXED GRAPHS

In this section, we define the concept of fixing neighborhood and fixed neighborhood. We also define the fixed graph using fixing neighborhood of graph.

A vertex v is *fixed* by an automorphism $\pi \in \Gamma(G)$ if $\pi(v) = v$. A vertex x is said to be fixed vertex in G if $\pi(x) = x$, $\forall \pi \in \Gamma(G)$. A vertex $x \in V(G)$ is said to *fix a pair* $(u, v) \in V(G) \times V(G)$, if $O_x(u) \neq O_x(v)$ in G . For $(u, v) \in V(G) \times V(G)$ and the set $F(u, v) = \{x \in V(G) : O_x(u) \neq O_x(v)\}$ is called the *fixing neighborhood* of (u, v) . For each $x \in V(G)$, the set $F(x) = \{(u, v) \in V(G) \times V(G) : O_x(u) \neq O_x(v)\}$ is called the *fixed neighborhood* of x . For any two distinct vertices u and v in G with $O(u) \neq O(v)$, $F(u, v) = V(G)$. Note that $fix(G) = 0$ if and only if G is rigid graph [15]. If G is rigid graph then for all distinct vertices $u, v \in V(G)$, $F(u, v) = V(G)$ but converse is not true. For example, consider an even path P_n on n vertices, then $F(u, v) = V(G)$ for any two distinct vertices of P_n which is not rigid.

For two distinct vertices u and v in a graph G , define $\iota_{u,v} : V(G) \rightarrow V(G)$,

$$\iota_{u,v}(x) = \begin{cases} v, & \text{if } x = u, \\ u, & \text{if } x = v, \\ x, & \text{otherwise.} \end{cases}$$

Then $\iota_{u,v}$ is an automorphism of G if and only if u and v are twins. Hence, we have the following result;

Lemma 2.1. *Let u and v be distinct vertices in a graph G . Then $\{u, v\} \subseteq F(u, v)$. Moreover, we have $F(u, v) = \{u, v\}$ if and only if u and v are twins.*

For two distinct vertices u and v in G , $R(u, v) = \{x \in V(G) : d(x, u) \neq d(x, v)\}$ where $d(x, u)$ is the distance between x and u .

Lemma 2.2. *Let u and v be two distinct vertices in G . Then $R(u, v) \subseteq F(u, v)$.*

Proof. If $x \in R(u, v)$, then $d(x, u) \neq d(x, v)$, which implies that $O_x(u) \neq O_x(v)$ since automorphisms preserve the distances in the graph G . Hence, the required result follows. \square

Note that in order to destroy automorphisms, only those vertices $u, v \in V(G)$ are of interest for which $O(u) = O(v)$ and $|O(u)| \geq 2$. So it is sufficient to consider $V_a(G)$ instead of $V(G) \times V(G)$. If S is a fixing set, then it is clear that $S \cap F(u, v) \neq \emptyset$ for any pair $(u, v) \in V_a(G)$. Moreover, for each pair in $V_a(G)$

can be fixed by elements of $A(G)$ only. If $F(u, v) = A(G)$, for all $u, v \in A(G)$, then $fix(G) = 1$ but converse is not true. To see this, consider the cartesian product of P_4 and P_5 , denoted by $P_4 \square P_5$ and $V(P_4 \square P_5) = \{u_{ij} | 1 \leq i \leq 4, 1 \leq j \leq 5\}$. Note that $A(P_4 \square P_5) = V(P_4 \square P_5)$ and $fix(P_4 \square P_5) = 1$. But $F(u_{11}, u_{15}) \neq A(P_4 \square P_5)$ because u_{13} does not fix the pair (u_{11}, u_{15}) .

The *fixed graph*, $I(G)$, of a graph G is a bipartite graph with bipartition $(V(G), V_a(G))$ and a vertex $x \in V(G)$ is adjacent to a pair $(u, v) \in V_a(G)$ if $x \in F(u, v)$. For a set $D \subseteq A(G)$, $N_{I(G)}(D) = \{(u, v) \in V_a(G) : x \in F(u, v) \text{ for some } x \in D\}$. In the fixed graph, $I(G)$, the minimum cardinality of a subset D of $V(G)$ such that $N_{I(G)}(D) = V_a(G)$ is the fixing number of G . For a graph G of order n , if $C(G) = V(G) \setminus A(G)$ and $F(u, v) = A(G)$ for all $u, v \in A(G)$, then $I(G) = K_{|A(G)|, |V_a(G)|} \cup \overline{K_{|C(G)|}}$. For a path P_n on even n vertices, $I(G) = K_{n, \frac{n}{2}}$ and for a path P_n on odd n vertices, $I(G) = K_{n-1, \frac{n-1}{2}} \cup K_1$. Least positive integer k such that every k -set of vertices of a graph G is a fixing set of G is called the *fixed number* of G denoted by $fxd(G)$. A graph G is said to be a k -fixed graph if $fix(G) = fxd(G) = k$. Javaid *et al.* studied fixed number of graphs in [21]. Lower and upper bounds on the cardinality of edge set of $I(G)$ for a k -fixed graph G were given in [19] and [21].

Proposition 2.3. *If G is a k -fixed graph of order $n \geq 2$, fixing number k and $|A(G)| = l$, then*

$$\frac{l}{2}(l - k + 1) \leq |E(I(G))| \leq n \binom{n}{2} - k + 1.$$

Rest of this section is devoted to the formulation of fractional version of the fixing number of a graph and its integer programming formulation.

Suppose $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V_a(G) = \{s_1, s_2, \dots, s_r\}$, $r \geq 1$. Let $B = (b_{ij})$ be the $r \times n$ matrix with

$$b_{ij} = \begin{cases} 1, & \text{if } s_i v_j \in E(I(G)), \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \leq i \leq r$ and $1 \leq j \leq n$.

The integer programming formulation of the fixing number is given by:

$$\begin{aligned} & \text{Minimize } f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n, \\ & \text{subject to the constraints} \\ & Bx \geq [1]_r \quad \text{and} \quad x_i \in \{0, 1\} \end{aligned}$$

where $x = [x_1, x_2, \dots, x_n]^T$, $[1]_k$ is the $k \times 1$ matrix all of whose entries are 1, and $[0]_n$ is the $n \times 1$ matrix all of whose entries are 0.

If we relax the condition, $x_i \in \{0, 1\}$ for every i and require that $x_i \geq 0$ for all i , then we obtain the following linear programming problem:

$$\begin{aligned} & \text{Minimize } f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n \\ & \text{subject to the constraints} \\ & Bx \geq [1]_r \quad \text{and} \quad x \geq [0]_n. \end{aligned}$$

In terms of the fixed graph $I(G)$ of G , solving this linear programming problem amounts to assigning non-negative weights to the vertices in $V(G)$ so that for each pair in $V_a(G)$, the sum of weights in its neighborhood is at least 1 and such that the sum of weights of the vertices of G is as small as possible. The smallest value for f is called the *fractional fixing number* of G .

Definition 2.4. Let G be a connected graph of order n . A function $g : V(G) \rightarrow [0, 1]$ is a *fixing function* FF of G if $g(F(u, v)) \geq 1$ for any pair $(u, v) \in V_a(G)$, where $g(F(u, v)) = \sum_{x \in F(u, v)} g(x)$ and $|g| = \sum_{v \in V} g(v)$. The *fractional fixing number*, denoted by $fix_f(G)$, is the minimum value of FF .

Definition 2.5. Let G be a connected graph of order n . A function $g : V(G) \rightarrow [0, 1]$ is a *resolving function* RF of G if $g(R(u, v)) \geq 1$ for two distinct vertices $u, v \in V(G)$, where $g(R(u, v)) = \sum_{x \in R(u, v)} g(x)$ and $|g| = \sum_{v \in V} g(v)$. The *fractional metric dimension*, denoted by $dim_f(G)$, is the minimum value of RF .

In the next theorem, we show that $dim_f(G)$ is an upper bound of $fix_f(G)$.

Theorem 2.6. For any connected graph G , we have $fix_f(G) \leq dim_f(G)$.

Proof. If G is a rigid graph, the $fix_f(G) = 0 \leq dim_f(G)$. Now suppose that G is a non-rigid graph. By Lemma 2.2, each resolving function of G is a fixing function of G . Therefore, our desired inequality holds. \square

3. CHARACTERIZATION OF GRAPHS WITH $fix_f(G) = \frac{|V(G)|}{2}$

In this section, we characterize the graphs having $fix_f(G) = \frac{|V(G)|}{2}$. For graphs with $fix(G) = 1$, it follows that $fix_f(G) = 1$ because the characteristic function of a minimal fixing set is an FF of G , it follows that $1 \leq fix_f(G) \leq fix(G) \leq fix^+(G) \leq n - 1$. Note that fixing function plays an important role while finding fractional fixing number of a graph. To define fixing function that meets all conditions, we need to know cardinalities of fixing neighborhoods of $(u, v) \in V_a(G)$. For a graph G of order n , we define

$$f(G) = \min\{|F(u, v)| : (u, v) \in V_a(G)\}.$$

Now, we express the fractional fixing number of G in terms of $f(G)$ in the following proposition:

Proposition 3.1. *Let G be a connected graph of order n . Then $\text{fix}_f(G) \leq \frac{n}{f(G)}$.*

Proof. Let $g : V(G) \rightarrow [0, 1]$ defined by $g(v) = \frac{1}{f(G)}$. For any two distinct vertices u and v , we have $g(F(u, v)) = \frac{|F(u, v)|}{f(G)} \geq 1$. Clearly g is a fixing function of G . Hence, $\text{fix}_f(G) \leq |g| = \frac{n}{f(G)}$. \square

By above proposition and Lemma 2.1, we have the following result:

Corollary 3.2. *For a connected graph G of order n , we have $\text{fix}_f(G) \leq \frac{n}{2}$.*

In the rest of this section, we characterize all graphs G attaining the upper bound in Corollary 3.2.

Lemma 3.3. *For a non-rigid graph G , we have*

$$\text{fix}_f(G) \leq \frac{|V(G)| - |C(G)|}{2},$$

Proof. Define a function $g : V(G) \rightarrow [0, 1]$,

$$g(x) = \begin{cases} 0, & \text{if } x \in C(G), \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Note that $|V(G)| - |C(G)| \geq 2$. Pick any two distinct vertices u and v in G . If $O(u) \neq O(v)$, then $F(u, v) = V(G)$, and so $g(F(u, v)) = |g| \geq 1$. If $O(u) = O(v)$, then $g(u) = g(v) = \frac{1}{2}$, which implies that $g(F(u, v)) \geq 1$ by Lemma 2.1. It follows that g is a fixing function. Hence, the desired result holds. \square

Given a graph H and a family of graphs $\mathcal{I} = \{I_v\}_{v \in V(H)}$, indexed by $V(H)$, their *generalized lexicographic product*, denoted by $H[\mathcal{I}]$, is defined as the graph with the vertex set $V(H[\mathcal{I}]) = \{(v, w) | v \in V(H) \text{ and } w \in V(I_v)\}$ and the edge set $E(H[\mathcal{I}]) = \{((v_1, w_1), (v_2, w_2)) | \{v_1, v_2\} \in E(H), \text{ or } v_1 = v_2 \text{ and } \{w_1, w_2\} \in E(I_{v_1})\}$.

Theorem 3.4. *Let G be a non-trivial graph of order n . Then the following conditions are pairwise equivalent.*

- (i) $\text{fix}_f(G) = \frac{n}{2}$.
- (ii) Each vertex in G has a twin.
- (iii) There exist a graph H and a family of graphs $\mathcal{I} = \{I_v\}_{v \in V(H)}$, where I_v is a non-trivial null graph or a non-trivial complete graph, such that G is isomorphic to $H[\mathcal{I}]$.

Proof. We show that (i) indicates (ii), (ii) indicates (iii), and (iii) indicates (i). Suppose (i) holds. Then $C(G) = \emptyset$ by Lemma 3.3. If there exists a vertex u in G such that u does not have a twin, then the following function $g : V \rightarrow [0, 1]$,

$$g(x) = \begin{cases} 0, & \text{if } x = u, \\ \frac{1}{2}, & \text{if } x \neq u, \end{cases}$$

is a fixing function of G by Lemma 2.1, which implies that $\text{fix}_f(G) \leq \frac{n-1}{2}$, a contradiction. So (ii) holds.

Suppose (ii) holds. For $x, y \in V(G)$, define $u \equiv v$ if and only if $x = y$ or x, y are twins. It is clear that \equiv is an equivalence relation. Suppose

$$O_1, \dots, O_m$$

are the equivalence classes. Then the induced subgraph on each O_i , denoted also by I_{O_i} , is a non-trivial null graph or a non-trivial complete graph. Let H be the graph with the vertex set $\{O_1, \dots, O_m\}$, where two distinct vertices O_i and O_j are adjacent if there exist $x \in O_i$ and $y \in O_j$ such that x and y are adjacent in G . It is routine to verify that G is isomorphic to $H[\mathcal{I}]$, where $\mathcal{I} = \{I_{O_i} : i = 1, \dots, m\}$. So (iii) holds.

Suppose (iii) holds. For $v \in V(H)$, write

$$V(I_v) = \{w_v^1, \dots, w_v^{s(v)}\}.$$

Then $s(v) \geq 2$, and (v, w_v^i) and (v, w_v^j) are twins in $H[\mathcal{I}]$, where $1 \leq i < j \leq s(v)$. Let h be a fixing function of $H[\mathcal{I}]$ with $|h| = \text{fix}_f(H[\mathcal{I}])$. By Lemma 2.1, we get

$$h((v, w_v^i)) + h((v, w_v^j)) \geq 1 \quad \text{for } 1 \leq i < j \leq s(v),$$

which implies that

$$\sum_{k=1}^{s(v)} h(v, w_v^k) \geq \frac{s(v)}{2},$$

and so

$$\text{fix}_f(G) = \text{fix}_f(H[\mathcal{I}]) = |h| = \sum_{v \in V(H)} \sum_{k=1}^{s(v)} h((v, w_v^k)) \geq \sum_{v \in V(H)} \frac{s(v)}{2} = \frac{|V(H[\mathcal{I}])|}{2} = \frac{n}{2}.$$

So (i) holds. We accomplish the proof. \square

For following families of graphs, each vertex in these graph has a twin. Using Theorem 3.4, we get the fractional fixing number of these families of graphs:

Example 3.5. $\text{fix}_f(G) = \frac{|V(G)|}{2}$ for each of the following graphs:

(1) $G = K_n, n \geq 2$.

- (2) $G = K_n - e$, where $n \geq 4$ and e is an arbitrary edge of K_n .
- (3) $G = K_{2t} - M$, $t \geq 2$ and M is a perfect matching in K_{2t} .
- (4) G is complete k -partite graph K_{n_1, n_2, \dots, n_k} , where $k \geq 2$ and $n_i \geq 2$.

The *join graph* $G + H$ is the graph obtained from G and H by joining each vertex of G with every vertex of H . Note that, if each vertex in G_i has a twin for $i \in \{1, 2\}$, then each vertex in $G_1 + G_2$ has a twin. Hence, we have:

Corollary 3.6. *Let Θ denote the collection of all connected graphs G with $\text{fix}_f(G) = \frac{|V(G)|}{2}$. If $G_1, G_2 \in \Theta$, then $G_1 + G_2 \in \Theta$.*

Corollary 3.7. *If $\text{fix}_f(G) = \frac{n}{2}$, then $\text{fix}_f(G + \overline{K_k}) = \frac{n+k}{2}$, for $k \geq 2$.*

Theorem 3.8. *Any connected graph H can be embedded as an induced subgraph of a connected graph G with $\text{fix}_f(G) = \frac{|V(G)|}{2}$.*

Proof. Let $V(H) = \{u_1, u_2, \dots, u_n\}$. Consider the graph G formed from H by replacing each vertex u_i of H by u_{i_1} and u_{i_2} , and joining u_{i_s} to u_{j_t} , where $s, t \in \{1, 2\}$, whenever u_i and u_j are adjacent in H . Hence, u_{i_1} and u_{i_2} are twins in G , and so $\text{fix}_f(G) = \frac{|G|}{2}$, and H is an induced subgraph of G . \square

4. FRACTIONAL FIXING NUMBER OF SOME FAMILIES OF GRAPHS

In this section, we determine the fractional fixing number of some families of graphs. A graph G is *vertex-transitive* if its automorphism group $\Gamma(G)$ acts transitively on the vertex set. For any two vertices v and w in $V(G)$, Γ_v and Γ_w are isomorphic and index of Γ_v in $\Gamma(G)$ is equal to the order of $V(G)$. In the following result, we give the fractional fixing number of a vertex-transitive graph G in terms of the parameter $f(G)$.

Theorem 4.1. *Let G be a vertex-transitive graph, then $\text{fix}_f(G) = \frac{|V(G)|}{f(G)}$.*

Proof. Let $f(G) = p$. Then there exists a pair of distinct vertices $(u, v) \in V_a(G)$ such that $|F(u, v)| = p$. Suppose $F(u, v) = \{r_1, r_2, \dots, r_p\}$. For any automorphism α of G , $F(\alpha(u), \alpha(v)) = \{\alpha(r_1), \alpha(r_2), \dots, \alpha(r_p)\}$. Let h be a fixing function of G with $\text{fix}_f(G) = |h|$. Then

$$h(\alpha(r_1)) + h(\alpha(r_2)) + \dots + h(\alpha(r_p)) = h(F(\alpha(u), \alpha(v))) \geq 1,$$

which implies that

$$\sum_{\alpha \in \Gamma(G)} (h(\alpha(r_1)) + h(\alpha(r_2)) + \dots + h(\alpha(r_p))) \geq |\Gamma(G)|.$$

Since G is vertex transitive, we have

$$|\Gamma_{r_1}| \cdot |h| + |\Gamma_{r_2}| \cdot |h| + \dots + |\Gamma_{r_p}| \cdot |h| \geq |\Gamma(G)|$$

which implies that $fix_f(G) \geq \frac{|V(G)|}{p}$. By Proposition 3.1, we have the required result. \square

Since cycle C_n of order n is vertex transitive, therefore we have the following result:

Corollary 4.2. *For the cycle C_n , we have*

$$fix_f(C_n) = \begin{cases} \frac{n}{n-2}, & \text{if } n \text{ is even,} \\ \frac{n}{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

A non-trivial connected graph G is *distance-transitive* if given any two ordered pairs of vertices (u_1, v_1) and (u_2, v_2) such that $d(u_1, v_1) = d(u_2, v_2)$, there is an automorphism σ of G such that $(u_2, v_2) = (\sigma(u_1), \sigma(v_1))$.

Lemma 4.3. *Let u and v be two distinct vertices in a distance-transitive graph G . Then $R(u, v) = F(u, v)$.*

Proof. Note that all distance-transitive graphs are vertex-transitive. Then G is non-rigid. Take any $x \in F(u, v)$. Then $O_x(u) \neq O_x(v)$. If $d(x, u) = d(x, v)$, then there is an automorphism σ of G such that $(x, v) = (\sigma(x), \sigma(u))$, which implies that $v \in O_x(u)$, and so $O_x(u) = O_x(v)$, a contradiction. Hence, we have $F(u, v) \subseteq R(u, v)$. Therefore, we get the desired result by Lemma 2.2. \square

According to the above lemma, we get the following result immediately.

Theorem 4.4. *For a distance-transitive graph G , we have $fix_f(G) = dim_f(G)$.*

The *Hamming graph*, denoted by $H_{n,k}$, has the vertex set $\{(x_1, \dots, x_n) | 1 \leq x_i \leq k, 1 \leq i \leq n\}$, with two vertices being adjacent if they differ in exactly one co-ordinate. Let X be a set of size n , and let $\binom{X}{k}$ denote the set of all k -subsets of X . The *Johnson graph*, denoted by $J(n, k)$, has $\binom{X}{k}$ as the vertex set, where two k -subsets are adjacent if their intersection has size $k - 1$.

It is well-known that $H_{n,k}$ and $J(n, k)$ are distance-transitive. The fractional metric dimension of $H_{n,2}$ was computed in [1]. Feng et al. [11] compute $dim_f(H_{n,k})$ for $k \geq 3$ and $dim_f(J(n, k))$. Combining all these results and Theorem 4.4, we get

Corollary 4.5. *Let n and k be positive integers at least 2.*

$$(i) \ fix_f(H_{n,k}) = \begin{cases} 2, & \text{if } k = 2, \\ \frac{k}{2}, & \text{if } k \geq 3. \end{cases}$$

(ii) If $n \geq 2k$, then

$$\text{fix}_f(J(n, k)) = \begin{cases} 3, & \text{if } (n, k) = (4, 2), \\ \frac{35}{17}, & \text{if } (n, k) = (8, 4), \\ \frac{n^2-n}{2kn-2k^2}, & \text{otherwise.} \end{cases}$$

The next result is a generalization of Theorem 3.4. Note that for $C(G) = \emptyset$, the next theorem coincides with Theorem 3.4.

Theorem 4.6. *Let G be a connected graph of order n . Then $\text{fix}_f(G) = \frac{n-|C(G)|}{2}$ if and only if each vertex in $A(G)$ has a twin.*

Proof. By Lemma 3.3, $\text{fix}_f(G) \leq \frac{n-|C(G)|}{2}$. Let h be any fixing function of G . Then by Lemma 2.1, $h(u) + h(v) \geq 1$ for all $u, v \in A(G)$. Adding these $n - |C(G)|$ inequalities, we get $\text{fix}_f(G) \geq |h| \geq \sum_{i=1}^{n-|C(G)|} h(u_i) \geq \frac{n-|C(G)|}{2}$. Therefore, $\text{fix}_f(G) = \frac{n-|C(G)|}{2}$. Converse part of this theorem is straight forward from Theorem 3.4. \square

The *friendship graph* F_n can be constructed by joining n copies/blocks of the cycle graph C_3 with a common vertex.

Corollary 4.7. *For friendship graph F_n , $\text{fix}_f(F_n) = n$.*

Proof. For $1 \leq i \leq n$, let $H_i = (x, a_i, b_i, x)$ be the n blocks of F_n . Note that $C(F_n) = \{x\}$ and $F(a_i, b_i) = \{a_i, b_i\}$. By Lemma 2.1, a_i and b_i are twins. Hence by Theorem 4.6, $\text{fix}_f(F_n) = \frac{|V|-|C(F_n)|}{2} = n$. \square

The *fan graph* $F_{1,n}$ of order $n+1$ is defined as the join graph $K_1 + P_n$.

Corollary 4.8. *For fan graph $F_{1,n}$ with $n \geq 3$,*

$$\text{fix}_f(F_{1,n}) = \begin{cases} 2, & \text{if } n = 3, \\ 1, & \text{if } n \geq 4. \end{cases}$$

Proof. Note that each vertex in $F_{1,3}$ has a twin, so by Theorem 3.4, $\text{fix}_f(F_{1,3}) = 2$. Now, suppose $n \geq 4$. Take a vertex $u \in V(F_{1,n})$ of degree 2. Then for each $(x, y) \in V_a(F_{1,n})$, we have $u \in F(x, y)$. Since $F_{1,n}$ is not a rigid graph so one has $\text{fix}(F_{1,n}) = 1$, which implies that $\text{fix}_f(F_{1,n}) = 1$, as desired. \square

For $v \in V(G)$, $G-v$ is known as the vertex deleted subgraph of G obtained by deleting v from the vertex set of G along with its incident edges.

Proposition 4.9. *For a connected graph G , $\text{fix}_f(G) - 1 \leq \text{fix}_f(G-v)$, where v is a vertex of G .*

Proof. Let $g : V(G - v) \rightarrow V(G - v)$ be a fixing function of $G - v$ such that $\text{fix}_f(G - v) = |g|$. Now, the function $g' : V(G) \rightarrow V(G)$ defined by

$$g'(u) = \begin{cases} g(u), & \text{if } u \neq v, \\ 1, & \text{if } u = v. \end{cases}$$

is a fixing function of G and hence $\text{fix}_f(G) \leq |g'|$. Thus $\text{fix}_f(G - v) = |g| = |g'| - 1 \geq \text{fix}_f(G) - 1$. \square

In the following result, the fractional fixing number of trees has been computed. Let $T = (V(T), E(T))$ be an n -vertex non-path tree with $n \geq 4$, then $\text{fix}_f(T) \geq 1$. A vertex of degree one is called a leaf.

Theorem 4.10. *The fractional fixing number of a tree T with n vertices satisfies the following statements:*

- (1) $0 \leq \text{fix}_f(T) \leq \frac{n-1}{2}$ and both bounds are tight.
- (2) Given $n, k \in \mathbb{N}$ with $2 \leq k \leq n - 1$ and $k \neq n - 2$, there exists a tree T of order n such that $\text{fix}_f(T) = \frac{k}{2}$.
- (3) A tree T such that $\text{fix}_f(T) = 0$ can only exist if $n = 1$ or $n \geq 7$.

Proof. (1) By definition, $\text{fix}_f(T) \geq 0$. Furthermore, T contains at most $n - 1$ leaves which implies, by Theorem 4.6, that fractional fixing number is at most $\frac{n-1}{2}$. Upper bound is sharp for star graph, $K_{1,n-1}$.
(2) Consider $n, k \in \mathbb{N}$ with $2 \leq k \leq n - 3$, a $u - v$ path P_{n-k} with group of leaves $\{v_1, v_2, \dots, v_k\}$ hanging from v . Theorem 4.6 implies its fractional fixing number is $\frac{k}{2}$. The star $K_{1,n-1}$ serves as example for $k = n - 1$.
(3) It was proved in [4] that a tree T such that $\text{fix}(T) = 0$ can only exist if $n = 1$ or $n \geq 7$. This implies $\text{fix}_f(T) = 0$ can exist only if $n = 1$ or $n \geq 7$. \square

There exists families of graphs for which for which $\dim_f(T)$ and $\text{fix}_f(T)$ are equal. Consider a tree T formed by connecting a single vertex u to k paths denoted by $P_m, P_{m+1}, \dots, P_{m+k-1}$ with lengths $m, m + 1, \dots, m + k - 1$, respectively. It is clear that such a tree is a rigid graph and $\text{fix}_f(T) = 0$ and it was shown in [22] that $\dim_f(T) = \frac{k}{2}$. Hence there exist graphs for which the difference between $\dim_f(T)$ and $\text{fix}_f(T)$ can be arbitrarily large.

Corollary 4.11. *For the wheel W_n , $n \geq 5$, we have*

$$\text{fix}_f(W_n) = \begin{cases} \frac{n}{n-2}, & \text{if } n - 1 \text{ is even,} \\ \frac{n}{n-3}, & \text{if } n - 1 \text{ is odd.} \end{cases}$$

Proof. Let $V(W_n) = \{u_1, u_2, \dots, u_{n-1}, u\}$ where u is the center of the wheel and $C_{n-1} = (u_1, u_2, \dots, u_{n-1})$, $n \geq 4$, is the rim. Since the center vertex u does not belong to $A(W_n)$, therefore the automorphism group of W_n is same as that of cycle C_{n-1} . Therefore, we have the required result. \square

In [1], it was shown that $\dim_f(W_n) = \frac{n-1}{4}$ for $n \geq 7$. Please note $\dim_f(W_n) - \text{fix}_f(W_n) \rightarrow \infty$ as $n \rightarrow \infty$.

5. FRACTIONAL FIXING NUMBER OF CORONA PRODUCT OF GRAPHS

Let G and H be two graphs with $|V(G)| = m$ and $|V(H)| = n$. *Corona product* of G and H , denoted by $G \odot H$, is the graph obtained from G and H by taking one copy of G and m copies of H and joining each vertex from the i^{th} -copy of H by an edge with the i^{th} -vertex of G . Let $u \in V(G)$ then H_u be the copy of H corresponding to the u -vertex of G . We write $H_u = \{(u, v) : v \in V(H)\}$ for $u \in V(G)$. For $x, y \in V(G \odot H)$, the fixing neighborhood of x, y is denoted by $F_{G \odot H}(x, y)$ and $F_G(x, y)$ denotes the fixing neighborhood of x, y in G .

Lemma 5.1. *Let G be a connected graph of order $m \geq 2$ and H be an arbitrary graph. Let $(x, y) \in V_a(G \odot H)$.*

- (1) *If $\{(x, y)\} \subseteq V_a(H_u)$ for some $u \in V(G)$, say $x = (u, v_1)$ and $y = (u, v_2)$, then*

$$F_{G \odot H}(x, y) = F_H(v_1, v_2).$$

- (2) *If $\{x, y\} \not\subseteq V_a(H_u)$ for any $u \in V(G)$, then there exists a vertex u_0 of G such that $H_{u_0} \subseteq F_{G \odot H}(x, y)$.*

Proof. (1) If $\{(x, y)\} \subseteq V_a(H_u)$ for any $u \in V(G)$, then it is clear that there exists $\alpha \in \Gamma_{V(G \odot H) \setminus H_u}$ such that $\alpha(x) = y$, therefore we have $F_{G \odot H}(x, y) \subseteq H_u$. Note that $(r, s) \in F_{G \odot H}(x, y)$ is equivalent to $s \in F_H(v_1, v_2)$. Hence, the desired result follows.

(2) Note that for $x \in V(G)$ and $y \in H_u$ or $y \in V(G)$ and $x \in H_u$, $(x, y) \notin V_a(G \odot H)$. Now, we have two cases:

Case 1: Let $(x, y) \in V_a(G)$, then for $\alpha \in \Gamma(G)$, $\alpha(x) = y$ if and only if $\alpha(H_x) = H_y$. This implies that H_x and $H_y \subseteq F_{G \odot H}(x, y)$.

Case 2: Let $x \in H_{u_1}$ and $y \in H_{u_2}$ for two distinct vertices $u_1, u_2 \in V(G)$. It is clear that by fixing any vertex of H_{u_1} or H_{u_2} , x cannot be mapped on y . Therefore, $H_{u_1} \subseteq F_{G \odot H}(x, y)$. \square

Fixing number of corona product of non-rigid graphs has been studied by Javaid *et al.* and they proved that $\text{fix}(G \odot H) = m \text{fix}(H)$ [20]. It is interesting

to note that a similar result is true for the fractional fixing number of $G \odot H$ as well.

Theorem 5.2. *Let G be a connected graph and H be a non-rigid graph with $|V(G)| = m \geq 2$. Then $\text{fix}_f(G \odot H) = m \text{fix}_f(H)$.*

Proof. Let g be a fixing function of $G \odot H$ with $|g| = \text{fix}_f(G \odot H)$. For each $u \in V(G)$, define $g_u : V(H) \rightarrow [0, 1]$ such that $v \mapsto g((u, v))$. For $(v_1, v_2) \in V_a(H)$, by Lemma 5.1,

$$g_u(F_H(v_1, v_2)) = \sum_{v \in F_H(v_1, v_2)} g((u, v)) = g(F_{G \odot H}((u, v_1), (u, v_2))) \geq 1,$$

which implies that

$$|g_u| \geq \text{fix}_f(H).$$

Since $V(H) \subseteq V(G \odot H)$, we have

$$|g| \geq \sum_{u \in V(G)} |g_u|.$$

Hence, $\text{fix}_f(G \odot H) \geq m \text{fix}_f(H)$. Now, we show that $\text{fix}_f(G \odot H) \leq m \text{fix}_f(H)$. Note that for any pair $(u, v) \in V_a(G)$, $H_u \subseteq F_{G \odot H}(u, v)$ and $H_v \subseteq F_{G \odot H}(u, v)$. Let $h : V(H) \rightarrow [0, 1]$ is a fixing function of H such that $|h| = \text{fix}_f(H)$. Define

$$h' : V(G \odot H) \rightarrow [0, 1], \quad w \mapsto \begin{cases} h(y), & \text{if } w = (x, y), \\ 0, & \text{if } w \in V(G). \end{cases}$$

Note that h' is a fixing function of $G \odot H$. Hence, $\text{fix}_f(G \odot H) \leq m \text{fix}_f(H)$ and the result follows. \square

Theorem 5.3. *Let G be a connected graph of order at least 2 and H be a rigid graph. Then*

$$\text{fix}_f(G \odot H) = \text{fix}_f(G).$$

Proof. If G is a rigid graph, then $G \odot H$ is a rigid graph, and so $\text{fix}_f(G \odot H) = 0 = \text{fix}_f(G)$. In the following, suppose that G is not a rigid graph. Then $G \odot H$ is not a rigid graph.

Let g be a fixing function of $G \odot H$ with $|g| = \text{fix}_f(G \odot H)$. Define

$$g' : V(G) \rightarrow [0, 1], \quad u \mapsto g(u) + \sum_{v \in H_u} g((u, v)).$$

For any $(u_1, u_2) \in V_a(G)$, we have

$$F_{G \odot H}(u_1, u_2) = F_G(u_1, u_2) \cup \bigcup_{u \in F_G(u_1, u_2)} H_u,$$

which implies that $g'(F_G(u_1, u_2)) = g(F_{G \odot H}(u_1, u_2)) \geq 1$, and so g' is a fixing function of G with $|g'| = |g|$. Therefore, one has $\text{fix}_f(G) \leq \text{fix}_f(G \odot H)$.

Let h be a fixing function of G with $|h| = \text{fix}_f(G)$. Define

$$h' : V(G \odot H) \rightarrow [0, 1], \quad u \mapsto \begin{cases} h(u), & \text{if } u \in V(G), \\ 0, & \text{otherwise.} \end{cases}$$

Note that H is a rigid graph. For $(x, y) \in V_a(G \odot H)$, we have $(x, y) \in V_a(G)$, or $(x, y) = ((u_1, v), (u_2, v))$ for some $((u_1, u_2), v) \in V_a(G) \times V(H)$. It follows that h' is a fixing function of $G \odot H$ with $|h'| = |h|$. Thus, we get $\text{fix}_f(G \odot H) \leq \text{fix}_f(G)$. Hence, the desired result follows. \square

Let H be a graph with maximum degree less than $|V(H)| - 1$. If H is a rigid graph, then $K_1 \odot H$ is a rigid graph, and so $\text{fix}_f(K_1 \odot H) = 0 = \text{fix}_f(H)$. If H is not a rigid graph, then $\text{fix}_f(K_1 \odot H) = \text{fix}_f(H)$ by a similar proof of Theorem 5.2. Consequently, we have

Theorem 5.4. *Let H be a graph with maximum degree less than $|V(H)| - 1$. Then*

$$\text{fix}_f(K_1 \odot H) = \text{fix}_f(H).$$

Now, we compute $\text{fix}_f(K_1 \odot H)$ if H has maximum degree $|V(H)| - 1$.

Theorem 5.5. *Let H be a graph with maximum degree $|V(H)| - 1$. Suppose the number of vertices with degree $|V(H)| - 1$ in H is k . Then*

$$\text{fix}_f(K_1 \odot H) = \begin{cases} \text{fix}_f(H) + 1, & \text{if } k = 1, \\ \text{fix}_f(H) + \frac{1}{2} & \text{if } k \geq 2. \end{cases}$$

Proof. Note that there are $k + 1$ vertices with degree $|V(H)|$ in $K_1 \odot H$ and all of them are twins. The rest of the proof is similar to the proof of Theorem 5.2. \square

6. FRACTIONAL FIXING NUMBER OF COMPOSITION PRODUCT OF GRAPHS

Let G and H be two graphs. The *composition product* of G and H , denoted by $G[H]$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G) \text{ and } v \in V(H)\}$, where (u, v) is adjacent to (x, y) whenever $ux \in E(G)$ or $u = x$ and $vy \in E(H)$. For any vertex $u \in V(G)$ and $v \in V(H)$, we define the vertex set $H(u) = \{(u, y) \in V(G[H]) : y \in V(H)\}$ and $G(v) = \{(x, v) \in V(G[H]) : x \in V(G)\}$.

Let G be a connected graph and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k with $|V(H_j)| \geq 2$ for each $j = 1, 2, \dots, k$. For any vertex $u \in V(G)$ and $1 \leq i \leq k$, we define the vertex set $H_i(u) = \{(u, v) \in V(G[H]) : v \in V(H_i)\}$. Let $|V(H_i)| = m_i$, $1 \leq i \leq k$. From the

definition of $G[H]$, it is clear that for every $(u, v) \in V(G[H])$, $\deg_{G[H]}(u, v) = \deg_G(u) \cdot |V(H)| + \deg_H(v)$. If G is a disconnected graph having $k \geq 2$ components G_1, G_2, \dots, G_k , then $G[H]$ is also a disconnected graph having k components such that $G[H] = G_1[H] \cup G_2[H] \cup \dots \cup G_k[H]$ and each component $G_i[H]$ is the composition product of connected component G_i of G with H , therefore throughout this section, we will assume G to be connected. For $x, y \in V(G[H])$, the fixing neighborhood of x, y is denoted by $F_{G[H]}(x, y)$. For a subgraph Q of a graph G , $\mathcal{F}(Q) = \{x \in V(G) : x \in F(u, v) \text{ for } (u, v) \in V_a(Q)\}$.

Lemma 6.1. [20] *Let G and H be two non-rigid graphs. For two distinct vertices $u, v \in V(G)$, if $z \in H(v)$, then $z \notin F_{G[H]}(x, y)$ for $(x, y) \in V_a(H(u))$.*

Lemma 6.2. [20] *Let $G[H]$ be the composition product of two non-rigid graphs G and H . Let H_1, H_2, \dots, H_k , $k \geq 1$, be the non-trivial components of H . Then for $u \in V(G)$ and $x \in H_j(u)$, $x \notin \mathcal{F}(H_i(u))$, $1 \leq i, j \leq k$ and $i \neq j$.*

Lemma 6.3. *Let G and H be two non-rigid graphs. Let H_1, H_2, \dots, H_k , $k \geq 1$, be the non-trivial components of H . Let $\{x, y\} \subseteq V_a(G[H])$.*

(1) *If $\{x, y\} \subseteq V_a(H_i(u))$, for some $u \in V(G)$, say $x = (u, v_1)$ and $y = (u, v_2)$, then $F_{G[H]}(x, y) = \{u\} \times F_{H_i}(v_1, v_2)$.*

(2) *If $\{x, y\} \subseteq V_a(G(v))$ for some $v \in V(H)$ say $x = \{u_1, v\}$ and $y = \{u_2, v\}$ then $F_{G[H]}(x, y) \supseteq F_G(u_1, u_2) \times \{v\}$.*

Proof. (1) holds from Lemma 6.1 and 6.2 and (2) is obvious. \square

Lemma 6.4. *Let G and H be two non-rigid graphs. Let H_1, H_2, \dots, H_k , $k \geq 1$, be the non-trivial components of H . Let g be a fixing function of $G[H]$ such that $\text{fix}_f(G[H]) = |g|$. Then for $u \in V(G)$, $g_i^u : V(H_i(u)) \rightarrow [0, 1]$, $(u, x) \mapsto g(u, x)$, is a fixing function of $H_i(u)$. Moreover, if h_i is a fixing function of H_i such that $\text{fix}_f(H_i) = |h_i|$ then $|g_i^u| \geq |h_i|$.*

Proof. For $(u, v_1), (u, v_2) \in V_a(H_i(u))$, by Lemma 6.3,

$$g_i^u(F_{H_i(u)}(u, v_1), (u, v_2)) = \sum_{w \in F_{H_i}(v_1, v_2)} g(u, w) = g(F_{G[H]}(u, v_1), (u, v_2)) \geq 1,$$

which implies that g_i^u is a fixing function of $H_i(u)$.

Suppose $|g_i^u| < |h_i|$. Then there exist two distinct vertices x, y in $H_i(u)$ such that $g_i^u(F_{H_i(u)}(x, y)) < 1$. Then $g_i^u(F_{H_i(u)}(x, y)) = g(F_{G[H]}(x, y)) < 1$, which contradicts the fact that g is a fixing function of $G[H]$. Hence, $|g_i^u| \geq |h_i|$. \square

Theorem 6.5. *Let G and H be two non-rigid graphs of orders m and n respectively. Let H_1, H_2, \dots, H_k , $k \geq 1$, be the non-trivial components of H . Then*

$$m \left(\sum_{i=1}^k \text{fix}_f(H_i) \right) \leq \text{fix}_f(G[H]) \leq \frac{mn}{2}.$$

Proof. Upper bound follows from Corollary 3.2. Let g be a fixing function of $G[H]$ with $|g| = \text{fix}_f(G[H])$. Then for $u \in V(G)$, $g_i^u : V(H_i(u)) \rightarrow [0, 1]$, $(u, x) \mapsto g(u, x)$, is a fixing function of $H_i(u)$ and $|g_i^u| \geq \text{fix}_f(H_i)$, by Lemma 6.4. Then $|g| \geq \sum_{u \in V(G)} \sum_{i=1}^k |g_i^u|$. Hence, $\text{fix}_f(G[H]) \geq m \left(\sum_{i=1}^k \text{fix}_f(H_i) \right)$. \square

Theorem 6.6. *Let G be a non-rigid and H a rigid graph. Then $\text{fix}_f(G[H]) = \text{fix}_f(G)$.*

Proof. For $(u_1, u_2) \in V_a(G)$, $F_{G[H]}((u_1, x), (u_2, x)) = F_G(u_1, u_2) \times V(H)$ for any $x \in V(H)$. Let g be a fixing function of $G[H]$ with $|g| = \text{fix}_f(G[H])$. Define $g' : V(G) \rightarrow [0, 1]$, $u \mapsto \sum_{v \in V(H)} g(u, v)$. Then

$$g'(F_G(u_1, u_2)) = \sum_{v \in V(H)} g(F_{G[H]}((u_1, v), (u_2, v))) \geq 1, \text{ which implies that } g' \text{ is}$$

a fixing function of G . Hence, $\text{fix}_f(G[H]) \geq \text{fix}_f(G)$.

Let h be a fixing function of G with $|h| = \text{fix}_f(G)$. Define

$$h' : V(G[H]) \rightarrow [0, 1], \quad w \mapsto \begin{cases} h(u), & \text{if } w = (u, v) \text{ for some fixed } v \in V(H), \\ 0, & \text{otherwise.} \end{cases}$$

For $(x, y) \in V_a(G[H])$, $x = (b, z)$ and $y = (c, z)$ for some $(b, c) \in V_a(G)$,

$h'(F_{G[H]}(x, y)) = h'(F_{G[H]}((b, z), (c, z))) = h(F_G(b, c)) \geq 1$, which implies that h' is a fixing function of $G[H]$. Hence, $\text{fix}_f(G) \geq \text{fix}_f(G[H])$. Hence, the desired result follows. \square

7. SUMMARY AND CONCLUSION

In this paper, the concept of the fractional fixing number of graphs has been studied. We have also introduced integer programming formulation of the fractional fixing number of graphs. Graphs with fractional fixing number $\text{fix}_f(G) = \frac{|V(G)|}{2}$ have also been characterized. The fractional fixing number of some families of graphs, corona product and composition product of graphs have also obtained. However, it remains to determine the fractional fixing number of several other families of graphs and graph products. Metric dimension and fixing number are closely related parameters. There are several graphs for which the study of the fractional fixing number of graphs is similar to that of the fractional metric dimension of graphs.

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